

Groupoid Quantization of Loop Spaces

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We review the various contexts in which quantized 2-plectic manifolds are expected to appear within closed string theory and M-theory. We then discuss how the quantization of a 2-plectic manifold can be reduced to ordinary quantization of its loop space, which is a symplectic manifold. We demonstrate how the latter can be quantized using groupoids. After reviewing the necessary background, we present the groupoid quantization of the loop space of \mathbb{R}^3 in some detail.

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1. Introduction

A symplectic manifold (M, ω) is a manifold M endowed with a closed and non-degenerate 2-form ω , i.e., $d\omega = 0$ and $\iota_X \omega = 0$ if and only if $X = 0$, where ι_X denotes contraction with the vector field $X \in \mathcal{C}^\infty(M, TM)$. The symplectic structure induces a Poisson structure on M according to the bracket

$$\{f, g\}_\omega := \iota_{X_f} \iota_{X_g} \omega, \quad (1.1)$$

where X_f denotes the Hamiltonian vector field of a smooth function f on M defined by $\iota_{X_f} \omega = df$. A symplectic manifold can therefore be used as a phase space in Hamiltonian mechanics. This is the starting point for geometric quantization [1].

Multisymplectic manifolds are categorified generalizations of symplectic manifolds: A p -plectic manifold (M, ϖ) is a manifold M endowed with a closed non-degenerate $p+1$ -form ϖ . Non-degeneracy means here again that $\iota_X \varpi = 0$, $X \in \mathcal{C}^\infty(M, TM)$, is equivalent to $X = 0$. Note that symplectic manifolds are 1-plectic in this nomenclature: p labels the level of categorification.

We will be mainly interested in 2-plectic manifolds, which come with a closed non-degenerate 3-form ϖ .

A 2-plectic form ϖ often induces a Nambu-Poisson 3-bracket [2, 3] (but not always, see [4] for a discussion). If this is the case, the underlying 2-plectic manifold can be used as a multiphase space in Nambu mechanics. Many approaches to the quantization of Nambu mechanics have been proposed over the years, see e.g. the references collected in [5]. Instead of focusing on the Nambu-Poisson structure and its quantization, we take a 2-plectic manifold (M, ϖ) as our starting point for quantization. We show that one can map the 2-plectic structure on M to a symplectic structure on the loop space of M by using a transgression map. We then quantize loop space using Hawkins' groupoid approach to quantization [6]. This article summarizes recent work which is a continuation of our studies in higher quantization [5, 7, 8, 9].

2. String theory and higher quantum geometry

There are a number of reasons why we are interested in 2-plectic manifolds and their quantization. As a motivation for our ensuing constructions, we start by reviewing some of the various contexts in which quantized 2-plectic manifolds are purported to appear and play a role in the dynamics of string theory and M-theory.

2.1 M2-M5 brane systems and Basu-Harvey equations

One of our main motivations stems from the description of a configuration of M-branes that arises from a lift of a D-brane configuration to M-theory. Consider a stack of k D1-branes extending along the x^6 -direction and ending on a D3-brane, which extends in the x^1, x^2, x^3 -directions:

$$\begin{array}{ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \text{D1} & \times & & & & & & \times \\
 \text{D3} & \times & \times & \times & \times & & &
 \end{array} \tag{2.1}$$

This BPS configuration gives a string theory realisation of magnetic monopoles [10, 11]: From the perspective of the D3-branes, the endpoints of the D1-branes appear as sources of magnetic flux and the dynamics of this system is described by the Bogomolny monopole equations. On the other hand, we can also study this system from the perspective of the D1-branes. Here the system is described in terms of three $\mathfrak{u}(k)$ -valued scalar fields X^i , $i = 1, 2, 3$, which parametrize the fluctuations of the D1-branes along the directions parallel to the worldvolume of the D3-branes. The dynamics is governed by the Nahm equations

$$\frac{d}{dx^6} X^i + \varepsilon^{ijk} [X^j, X^k] = 0. \tag{2.2}$$

A simple solution is given by $X^i = \frac{1}{x^6} G^i$, where $G^i = \varepsilon^{ijk} [G^j, G^k]$ generate a k -dimensional irreducible representation of $\mathfrak{su}(2)$. This is called a fuzzy funnel [12]: any given point x^6 of the worldvolume of the D1-branes polarizes into a fuzzy sphere with radius $\frac{1}{x^6}$. The constant matrices G^i are here interpreted as spherical coordinates on this fuzzy sphere. This polarization provides the transition between the one-dimensional worldvolume of the D1-branes and the three-dimensional worldvolume of the D3-branes.

The configuration (2.1) can be lifted to M-theory by first T-dualizing along the x^5 -direction and subsequently interpreting x^4 as the M-theory direction. The resulting configuration

$$\begin{array}{ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \text{M2} & \times & & & & & \times & \times \\
 \text{M5} & \times & \times & \times & \times & \times & \times &
 \end{array} \tag{2.3}$$

yields a self-dual string [13] from the perspective of the M5-brane. The perspective of the M2-branes is described by the Basu-Harvey equations [14]

$$\frac{d}{dx^6} X^\mu + \varepsilon^{\mu\nu\kappa\lambda} [X^\nu, X^\kappa, X^\lambda] = 0, \tag{2.4}$$

where $\mu, \nu, \dots = 1, \dots, 4$ and the scalar fields X^μ now take values in a 3-Lie algebra: A 3-Lie algebra is an extension of the notion of a Lie algebra to a vector space endowed with a totally antisymmetric trilinear bracket, which also satisfies a corresponding higher version of the Jacobi identity. Here the simple solution corresponding to a fuzzy funnel reads as $X^\mu = \frac{1}{\sqrt{2x^6}} G^\mu$, where $G^\mu = \varepsilon^{\mu\nu\kappa\lambda} [G^\nu, G^\kappa, G^\lambda]$. Clearly we would like to have an interpretation of this solution as a polarization of points of the worldvolume of the M2-branes into fuzzy three-spheres S^3 with radius $\frac{1}{\sqrt{2x^6}}$. Unfortunately, no consistent such interpretation exists in the literature to our knowledge.

Just as the area form turns S^2 into a symplectic manifold, the volume form of S^3 turns the three-sphere into a 2-plectic manifold. A comprehensive approach to the quantization of 2-plectic manifolds should therefore be able to fill this gap in the interpretation of the fuzzy M2-brane funnel.

2.2 Quantum geometry of M5-branes

The worldvolume field theory of a flat D-brane receives noncommutative Moyal-type deformations if a constant B -field background of 10-dimensional supergravity is turned on. Consider, e.g., the background 2-form field $B = b dx^1 \wedge dx^2$ on a D3-brane extending in the x^1, \dots, x^3 -directions. This induces the Moyal-type deformation of the coordinate algebra of the D3-brane given by [15, 16]

$$[x^1, x^2] = i \frac{\lambda^2}{1 + \lambda^2} \stackrel{\lambda \ll 1}{\sim} i \lambda^2, \tag{2.5}$$

where $\lambda = 2\pi \alpha' b$. This commutation relation describes a Heisenberg algebra which may be regarded as a quantization of the symplectic structure on \mathbb{R}^2 provided by the B -field. In [17] it is demonstrated how the Nahm equations can be used to encode boundary conditions for open strings; the commutation relation (2.5) then accounts for the induced shift in open string boundary conditions due to a background B -field.

Analogously, turning on a constant 3-form C -field background of 11-dimensional supergravity should yield quantum deformations of the worldvolume geometry of an M5-brane. These deformations can be understood as accounting for the modification in boundary conditions for open membranes ending on the M5-brane in the Basu-Harvey equations [17]. For example, it was found that the constant C -field background

$$C = \theta dx^0 \wedge dx^1 \wedge dx^2 + \theta' dx^3 \wedge dx^4 \wedge dx^5 \tag{2.6}$$

gives rise to 3-Lie algebra-type deformations of the coordinate algebra of an M5-brane extending in the x^1, \dots, x^5 -directions with

$$[x^0, x^1, x^2] = i\theta \quad \text{and} \quad [x^3, x^4, x^5] = i\theta'. \quad (2.7)$$

These relations define a pair of Nambu-Heisenberg algebras. Again the C -field (2.6) provides a 2-plectic structure on $\mathbb{R}^{1,2} \times \mathbb{R}^3$, and the appropriate worldvolume deformations should be obtained from a quantization of this 2-plectic manifold.

The effect of a constant C -field on an M2-brane ending on an M5-brane was also studied some time ago in [18, 19] from the perspective of loop space: quantization of the open membranes then yields a noncommutative loop space structure, analogously to the way in which quantization of open strings ending on a D-brane in a constant B -field induces a noncommutative coordinate algebra. We will come back to these results later on, thus clarifying the precise manner in which the 3-Lie algebra structure (2.7) is meant to be a repackaging of the complicated noncommutative loop space structure [17].

2.3 T-duality and nonassociative closed string geometry

Quantized 2-plectic structures have also recently emerged in the context of *closed* strings.

Firstly, it has been suggested that the phase space of the bosonic string is a 2-plectic manifold [20]. A quantization of the bosonic string would therefore correspond to a quantization of the underlying 2-plectic manifold. Instead of quantizing a 3-bracket, the authors of [20] propose to quantize a bracket on certain one-forms defined by

$$\{\alpha, \beta\}_{\overline{\omega}}^{(1)} := \iota_{X_\alpha} \iota_{X_\beta} \overline{\omega} \quad (2.8)$$

where $\alpha, \beta \in \Omega^1(M)$ are “Hamiltonian one-forms” in the sense that there exists vector fields X_α and X_β on M such that $\iota_{X_\alpha} \overline{\omega} = d\alpha$ and $\iota_{X_\beta} \overline{\omega} = d\beta$. This bracket is skew-symmetric, but the Jacobi identity is violated by a term proportional to $d\iota_{X_\alpha} \beta$. The resulting algebra of one-forms thus has the structure of a Lie 2-algebra. Our loop space quantization will resolve both the conceptual problem of quantizing one-forms and the failure of the Jacobi identity.

Secondly, closed string 3-form flux backgrounds have recently been used to derive nonassociative spacetime structures. For example, applying three T-dualities to the closed string mode expansions on a three-torus $M = T^3$ with constant 3-form H -flux leads to a modification of the phase space commutation relations given by [21]

$$[x^i, x^j] = i\theta^{ijk} p_k, \quad [x^i, p_j] = i\delta_j^i \quad \text{and} \quad [p_i, p_j] = 0. \quad (2.9)$$

This bracket defines a pre-Lie algebra structure but not a Lie algebra, as the Jacobi identity is not satisfied: the Jacobiator defines a nonassociative 3-bracket structure $[x^i, x^j, x^k] = i\theta^{ijk}$. It should also be properly understood in a categorified framework as a (semi-strict) Lie 2-algebra. Such nonassociative structures have been confirmed through an analysis of closed string three-point functions in the $SU(2)$ WZW model [22], and in an H -linearized expansion of conformal field theory on flat space [23] in both geometric and non-geometric 3-form flux backgrounds. It raises the tantalizing possibility of deriving a theory of noncommutative and nonassociative gravity from non-geometric flux compactifications.

The appearance of nonassociativity in H -flux backgrounds is not new and was observed some time ago in the context of open string theory [24], where it was found that open string two-point functions reproduce the noncommutative and nonassociative Kontsevich star-product for twisted Poisson structures. We can understand the loss of associativity in the following way. Just like the commutator is a quantization of a Poisson bracket which encodes the failure of a product to be commutative, the quantization of a Nambu-Poisson 3-bracket should encode the failure of a product to be associative. This suggests that appropriately quantized 2-plectic manifolds do not give rise to ordinary Hilbert spaces and linear operator algebras on them, but to some more general structures. In the following we will describe an attempt to understand these structures in the context of loop space quantization; this is very natural from the point of view of closed strings.

That there does still exist a suitable C^* -algebraic framework can be seen by understanding the relation to *topological* nonassociative tori [25, 26, 27]. The general setting is where the target space is a principal torus bundle

$$M \xrightarrow{T^d} W \quad (2.10)$$

of rank d with quantized three-form H -flux $[H] \in H^3(M, \mathbb{Z})$. This background and its duals are neatly described in an algebraic language: There is a continuous trace C^* -algebra \mathcal{A} with Dixmier-Douady class $[H]$ whose spectrum is X . The T -dual algebra which arises from dualizing along the fibre directions is then a crossed-product $\widehat{\mathcal{A}} = \mathcal{A} \rtimes_{\alpha} \widehat{\mathbb{R}^d}$ of \mathcal{A} with the Pontrjagin dual of the abelian group \mathbb{R}^d along the action $\alpha : \mathbb{R}^d \rightarrow \text{Aut}(\mathcal{A})$ induced by translations in the fibres. The algebra $\widehat{\mathcal{A}}$ is not a continuous trace algebra in general and so does not have a well-defined geometric spectrum; hence it gives a “global” description of some of the recent non-geometric backgrounds which have appeared in string theory.

To elucidate this description, let us consider the simplest case where $M = T^3$ is a trivial T^d -bundle over $W = T^{3-d}$. For $d = 0$ this is just the three-torus T^3 , which we endow with H -flux and B -field given by

$$H = k dx^1 \wedge dx^2 \wedge dx^3 = dB \quad \text{and} \quad B = k x^1 dx^2 \wedge dx^3, \quad k \in \mathbb{Z}. \quad (2.11)$$

In the doubled geometry the nonassociativity is manifested in the dual coordinates \tilde{x}^i as [21, 23]

$$[\tilde{x}^2, \tilde{x}^3] \sim w_1 \quad \text{and} \quad [\tilde{x}^1, \tilde{x}^2, \tilde{x}^3] \neq 0, \quad (2.12)$$

where w_i are winding modes. We now describe each of the three T -duals in turn:

- For $d = 1$, T -dualising along the x^3 -direction yields a continuous trace C^* -algebra whose spectrum is the Heisenberg nilmanifold, i.e., the quotient of the three-dimensional Heisenberg group $H_{\mathbb{R}}$ by a cocompact lattice $H_{\mathbb{Z}}$. The nonassociativity structure becomes

$$[\tilde{x}^2, x^3] \sim w_1 \quad \text{and} \quad [\tilde{x}^1, \tilde{x}^2, x^3] \neq 0. \quad (2.13)$$

- For $d = 2$, performing an additional T -duality transformation along the x^2 -direction yields the dual algebra $\widehat{\mathcal{A}} = C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$, where \mathcal{K} is the algebra of compact operators and $C^*(H_{\mathbb{Z}})$ is the convolution C^* -algebra of the lattice $H_{\mathbb{Z}}$ regarded as an abelian group. This algebra

describes a fibration of noncommutative two-tori T_θ^2 over S^1 with deformation parameter $\theta = kx^1$ induced by the B -field and varying over the base. This is the global description of a T -fold, i.e., a locally Riemannian space whose transition functions between coordinate patches involve T-duality transformations. The noncommutativity along the fibre directions appears in the linearized brackets

$$[x^2, x^3] \sim w_1 \quad \text{and} \quad [x^1, x^2, x^3] \neq 0. \quad (2.14)$$

- For $d = 3$, the additional T-duality along x^1 is strictly speaking forbidden as the B -field background is not invariant under translations around this direction. Nevertheless, considerations from string field theory suggest that the resulting background should be included when considering the totality of closed string backgrounds. Unlike the T-fold, this background does not admit even a local description as a Riemannian manifold, and is called a non-geometric R -flux background. Algebraically, the dual algebra $\widehat{\mathcal{A}} = \mathcal{K}(L^2(\widehat{T}^3)) \rtimes_{u_\phi} \widehat{T}^3$ is called the nonassociative three-torus T_ϕ^3 . It is described by a convolution product which is twisted by a 3-cocycle $\phi \in Z^3(\widehat{T}^3, \text{U}(1))$ associated to the 3-form H . The linearized nonassociativity structure is encoded in the brackets

$$[x^2, x^3] \sim p_1 \quad \text{and} \quad [x^1, x^2, x^3] \neq 0. \quad (2.15)$$

This nonassociative algebra may be regarded as a C^* -algebra in a monoidal category [28]; it may thus be identified as the convolution C^* -algebra of the Lie 2-group which integrates the Lie 2-algebra (2.9).

3. Quantization of symplectic manifolds

We will now review two well-known quantization prescriptions, which are both combined in the groupoid approach to quantization. First, we recall geometric and Berezin-Toeplitz quantization using the example of the complex projective line $\mathbb{C}P^1$. Second, we review the quantization of the dual of a Lie algebra using the example of κ -Minkowski space.

3.1 Berezin-Toeplitz quantization

The construction of the Hilbert space in Berezin-Toeplitz quantization is identical to that of geometric quantization. It is only in the construction of the quantization maps that the two prescriptions differ.

We call a symplectic manifold (M, ω) *quantizable* if there is a *prequantum line bundle* over M . A prequantum line bundle (L, h, ∇) consist of a line bundle $L \rightarrow M$ together with a hermitian metric h and a connection ∇ that is compatible with h . Moreover, we demand that the *quantization condition*

$$F_\nabla := \nabla^2 = -2\pi i \omega \quad (3.1)$$

is satisfied, which implies that $[\omega] \in H^2(M, \mathbb{Z})$. Note that a positive tensor power of a prequantum line bundle is again a prequantum line bundle. The datum (L, h, ∇) is called a *prequantization* of the symplectic manifold (M, ω) . The quantization condition (3.1) implies that L is positive or

ample. Therefore, the space of global sections of L is interesting enough to be identified with the Hilbert space \mathcal{H} in our quantization. It is well-known, however, that taking smooth sections of L yields a Hilbert space which is too large. To restrict the Hilbert space to the appropriate size, one introduces a *polarization*, i.e., a smooth integrable lagrangian distribution. We can then define the Hilbert space as the subspace of smooth global sections that are annihilated by the polarization. For us, it will be convenient to work with Kähler polarization, i.e., we regard M as a complex manifold and use the anti-holomorphic tangent bundle as a distribution. The resulting Hilbert space is thus given by global holomorphic sections of the prequantum line bundle L .

In the case (\mathbb{CP}^1, ω) , where ω is the usual Kähler form giving rise to the Fubini-Study metric on \mathbb{CP}^1 , we can use the positive tensor powers of the hyperplane line bundle $\mathcal{O}(1)$ (the dual of the tautological line bundle) as prequantum line bundles, i.e., we define $L_k := \mathcal{O}(k)$. The finite-dimensional Hilbert space $\mathcal{H}_k := H^0(\mathbb{CP}^1, L_k)$ can then be identified with the space of homogeneous polynomials of degree k in the homogeneous coordinates z_α , $\alpha = 1, 2$, on \mathbb{CP}^1 . This space is isomorphic to the k -particle Fock space in the Hilbert space of two harmonic oscillators with creation and annihilation operators \hat{a}_α^\dagger and \hat{a}_α satisfying $[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$ and $\hat{a}_\alpha|0\rangle = 0$:

$$\mathcal{H}_k \cong \text{span}_{\mathbb{C}}(z_{\alpha_1} \cdots z_{\alpha_k}) = \text{span}_{\mathbb{C}}(\hat{a}_{\alpha_1}^\dagger \cdots \hat{a}_{\alpha_k}^\dagger |0\rangle) . \quad (3.2)$$

To construct a map between functions on M and endomorphisms on \mathcal{H} , we use the Rawnsley coherent states. They form an overcomplete set of states labelled by points on M , and they can be constructed explicitly for any quantizable symplectic manifold. In the case of \mathbb{CP}^1 , they are the truncated Glauber coherent states

$$|z\rangle = \frac{1}{k!} (\bar{z}_\alpha \hat{a}_\alpha^\dagger)^k |0\rangle . \quad (3.3)$$

From these, we construct the *coherent state projector*

$$\mathcal{P}(z) := \frac{|z\rangle\langle z|}{\langle z|z\rangle} , \quad (3.4)$$

which is simultaneously an endomorphism on the Hilbert space \mathcal{H} and a smooth function on M . It therefore provides a bridge between the quantum and classical pictures, and we can use it to define quantization maps.

First, let us introduce the injective Berezin symbol

$$\sigma : \text{End}(\mathcal{H}) \longrightarrow \Sigma \subset \mathcal{C}^\infty(M) , \quad \sigma(\hat{f})(z) = \text{tr}_{\mathcal{H}}(\mathcal{P}(z)\hat{f}) . \quad (3.5)$$

The smooth functions in $\Sigma := \sigma(\text{End}(\mathcal{H}))$ form the set of quantizable functions, and we define a quantization map

$$\hat{\cdot} : \Sigma \longrightarrow \text{End}(\mathcal{H}) , \quad \hat{f} = \sigma^{-1}(f) . \quad (3.6)$$

Alternatively, we can introduce the Berezin-Toeplitz quantization map

$$T : \mathcal{C}^\infty(M) \longrightarrow \text{End}(\mathcal{H}) , \quad T(f) := \int_M d\mu(z) \mathcal{P}(z) f(z) , \quad (3.7)$$

where $d\mu(z)$ is the canonical Liouville measure on M .

In the case of the Riemann sphere \mathbb{CP}^1 , the functions in $\Sigma_k = \sigma(\text{End}(\mathcal{H}_k))$ are of the form¹

$$f(z) = \sum_{\alpha_i, \beta_i=1,2} f^{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_k} \frac{z_{\alpha_1} \dots z_{\alpha_k} \bar{z}_{\beta_1} \dots \bar{z}_{\beta_k}}{|z|^{2k}}, \quad (3.8)$$

where $f^{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_k} = (f^{\beta_1 \dots \beta_k \alpha_1 \dots \alpha_k})^* \in \mathbb{C}$, $|z|^2 := \bar{z}_\alpha z_\alpha$, and the quantization map reads explicitly as

$$f(z) \mapsto \hat{f} = \sum_{\alpha_i, \beta_i=1,2} f^{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_k} \frac{1}{k!} \hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_k}^\dagger |0\rangle \langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_k}. \quad (3.9)$$

The quantization map $\widehat{}$ therefore creates normal ordered operators. The quantization map T gives the corresponding anti-normal ordered operators. For further details on Berezin-Toeplitz quantization, see e.g. [29] and references therein.

3.2 Quantization of the dual of a Lie algebra

The groupoid approach to quantization is an extension of the well-known procedure of quantizing the dual of a Lie algebra, which we now briefly review.

We start from a Lie group G with Lie algebra \mathfrak{g} . There is a natural Poisson structure on \mathfrak{g}^* that is defined as follows: Linear functions on \mathfrak{g}^* can be identified with elements of \mathfrak{g} , and for those functions we define $\{g_1, g_2\}(x) := \langle x, [g_1, g_2] \rangle$, where $g_1, g_2 \in \mathfrak{g}$, $x \in \mathfrak{g}^*$ and $\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{C}$ denotes the dual pairing. Via the Leibniz identity, this Poisson bracket extends to polynomial functions on \mathfrak{g}^* , which in turn are dense in $\mathcal{C}^\infty(\mathfrak{g}^*)$.

To quantize the resulting Poisson algebra, we perform the following steps [30]: We first Fourier transform to obtain elements in $\mathcal{C}^\infty(\mathfrak{g})$. We then identify \mathfrak{g} with G in a neighborhood of the identity element via the exponential mapping. On G , we can use the convolution product between functions induced by the group multiplication and the Baker-Campbell-Hausdorff formula. We can then perform the inverse operations to pullback the result to $\mathcal{C}^\infty(\mathfrak{g}^*)$. For nilpotent Lie algebras, the exponential map between \mathfrak{g} and G is a global diffeomorphism. In this case, the above construction is equivalent to both Kontsevich's deformation quantization and quantization via the universal enveloping algebra of \mathfrak{g} [31].

To illustrate the procedure, let us use it to quantize d -dimensional κ -Minkowski space, which we identify with the dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} . The generators g^0, g^1, \dots, g^{d-1} of \mathfrak{g} are naturally identified as coordinate functions on \mathfrak{g}^* , and they have the Lie brackets

$$[g^0, g^i] = \frac{i}{\kappa} g^i \quad \text{and} \quad [g^i, g^j] = 0 \quad \text{for } 0 < i, j < d \quad (3.10)$$

with a constant $\kappa > 0$. The corresponding Lie group G is generated by $W(k_0, \vec{k}) = V_{\vec{k}} U_{k_0}$ with $U_{k_0} := \exp(i k_0 g^0)$ and $V_{\vec{k}} := \exp(-i \sum_i k_i g^i)$. The resulting group multiplication reads as

$$W(k_0, \vec{k}) W(k'_0, \vec{k}') = W(k_0 + k'_0, \vec{k} + e^{-k_0/\kappa} \vec{k}'). \quad (3.11)$$

We start by Fourier transforming a function $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ via

$$\tilde{f}(g) = \int_{\mathfrak{g}^*} d\mu_{\mathfrak{g}^*}(x) e^{-2\pi i \langle x, g \rangle} f(x), \quad (3.12)$$

¹The function space Σ_k is the linear span of the spherical harmonics $Y_{\ell m}$ on S^2 with $\ell \leq k$, $|m| \leq \ell$.

where $d\mu_{\mathfrak{g}^*}(x)$ denotes the invariant Haar measure on \mathfrak{g}^* regarded as an additive abelian group. We then identify \mathfrak{g} with G via the exponential mapping, and define the inverse Fourier transform according to

$$W(\tilde{f}) := \int_G d\mu_G(p_0, \vec{p}) W(p_0, \vec{p}) \tilde{f}(p_0, \vec{p}) , \quad (3.13)$$

where $d\mu_G(p_0, \vec{p}) = e^{p_0/\kappa} dp_0 d\vec{p}$ is the invariant Haar measure on G . The convolution product \circledast of two functions is defined via $W(\tilde{f})W(\tilde{f}') = W(\tilde{f} \circledast \tilde{f}')$, and here we obtain

$$(\tilde{f} \circledast \tilde{f}')(p_0, \vec{p}) = \int_G dp'_0 d\vec{p}' e^{p'_0/\kappa} \tilde{f}(p'_0, \vec{p}') \tilde{f}(p_0 - p'_0, e^{p'_0/\kappa}(\vec{p} - \vec{p}')) . \quad (3.14)$$

In a similar way one readily quantizes any Poisson vector space (\mathbb{R}^d, π) with constant Poisson bivector π , which yields the usual direct sums of Moyal planes and ordinary vector spaces.

4. Groupoid approach to quantization

4.1 From groups to groupoids

Any group may be regarded as a small category with one object $\mathbf{1}$, in which all morphisms, which correspond to the group elements, are invertible. Dropping the restriction that there is only one unit, we arrive at a groupoid: A *groupoid* is a small category in which every morphism is invertible. Analogously to Lie groups, we introduce *Lie groupoids*: Here both sets of objects and morphisms form manifolds and all morphisms are smooth.

Equivalently, we can describe a groupoid by a set B of units (called the base) and a set \mathcal{G} of morphisms or arrows between units (also called the groupoid) together with the following maps:

- The source and target maps $s, t : \mathcal{G} \rightrightarrows B$ that yield the tail and the head of an arrow.
- The object inclusion map $\mathbf{1} : B \hookrightarrow \mathcal{G}$, which yields an arrow starting and ending at the given unit and therefore satisfies $s(\mathbf{1}(x)) = t(\mathbf{1}(x)) = x$ for all $x \in B$.
- An associative partial multiplication $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ that concatenates two composable arrows. It is thus defined for arrows $(g, h) \in \mathcal{G} \times \mathcal{G}$ with $t(g) = s(h)$ (the “2-nerve” of \mathcal{G}), and we have $s(m(g, h)) = s(g)$ and $t(m(g, h)) = t(h)$, as well as $m(g, \mathbf{1}(t(g))) = g$ and $m(\mathbf{1}(s(g)), g) = g$.

Finally, we demand that each arrow $g \in \mathcal{G}$ has a two-sided inverse g^{-1} corresponding to the inverse arrow, i.e., $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $m(g, g^{-1}) = \mathbf{1}(s(g))$, and $m(g^{-1}, g) = \mathbf{1}(t(g))$. In a Lie groupoid, the sets B and \mathcal{G} are manifolds and the groupoid maps $s, t, \mathbf{1}, m$ are all smooth.

A trivial example of a groupoid is a group G , where B consists of a single point, \mathcal{G} is identified with G , the source and target maps are trivial, and the inclusion map of the point yields the identity element of G . A non-trivial example that will reappear in slightly altered form later on is that of the *pair groupoid*. Given a manifold M , we use points in M as units, $B = M$, and consider all possible arrows having their heads and tails in M , $\mathcal{G} = M \times M$. The identity map is thus $\mathbf{1}(x) = (x, x)$ and the other structure maps of the groupoid are defined by

$$x \xrightarrow{(x,y)} y \xrightarrow{(y,z)} z \quad (4.1)$$

for all $x, y, z \in M$.

The linearized version of a Lie group is a Lie algebra, which is given by the tangent space to the Lie group at the identity element. Because a Lie groupoid has more than one unit, one has to consider the union of all these tangent spaces. This gives rise to the Lie algebroid $\text{Lie}(\mathcal{G})$ of a Lie groupoid \mathcal{G} which is a vector bundle over B with total space

$$\text{Lie}(\mathcal{G}) = \bigcup_{x \in B} T_{1(x)}(t^{-1}(x)) \subset T\mathcal{G}. \quad (4.2)$$

More generally, a *Lie algebroid* \mathcal{E} is a vector bundle \mathcal{E} over a base manifold B endowed with a Lie bracket $[-, -]_{\mathcal{E}}$ on smooth sections of \mathcal{E} and a bundle morphism $\rho : \mathcal{E} \rightarrow TB$, called the *anchor map*, which is compatible with the Lie bracket on sections, i.e., the tangent map to ρ is a Lie algebra homomorphism,

$$d\rho_{[\psi_1, \psi_2]_{\mathcal{E}}} = [d\rho_{\psi_1}, d\rho_{\psi_2}]_{TB}, \quad \psi_1, \psi_2 \in \mathcal{C}^{\infty}(B, \mathcal{E}), \quad (4.3)$$

and a Leibniz rule is satisfied when multiplying sections of \mathcal{E} by smooth functions on B ,

$$[\psi_1, f\psi_2]_{\mathcal{E}} = f[\psi_1, \psi_2]_{\mathcal{E}} + \rho_{\psi_1}(f)\psi_2, \quad \psi_1, \psi_2 \in \mathcal{C}^{\infty}(B, \mathcal{E}), \quad f \in \mathcal{C}^{\infty}(B). \quad (4.4)$$

We thus have the fibrations

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho} & TB \\ & \searrow & \downarrow \\ & & B \end{array} \quad (4.5)$$

The Lie algebroid of a Lie groupoid which is also a Lie group is just the usual Lie algebra together with the trivial anchor map. In the case of the pair groupoid $\mathcal{G} = M \times M$, the corresponding Lie algebroid is the tangent bundle

$$\text{Lie}(M \times M) = \bigcup_{x \in M} x \times T_x M = TM. \quad (4.6)$$

Above we derived Lie algebroids from Lie groupoids. While the inverse operation is defined for Lie algebras, which we can integrate to Lie groups, the same is not true in general for Lie algebroids. We will encounter a criterion for integrability in the next subsection. For more details on groupoids and algebroids, see e.g. [32].

4.2 Summary of groupoid quantization

The starting point of any quantization is a Poisson manifold (M, π) , where π is the bivector field that encodes the Poisson bracket through $\{f, g\}_{\pi} := \pi(df, dg)$ for all $f, g \in \mathcal{C}^{\infty}(M)$. A Poisson manifold naturally comes with a Lie algebroid structure on its cotangent bundle T^*M : The anchor map $\rho : T^*M \rightarrow TM$ is defined as $\alpha \mapsto \pi(\alpha, -)$. The compatible Lie bracket on exact one-forms is given by $[df, dg]_{T^*M} = d\{f, g\}_{\pi}$, which can be extended via the Leibniz rule (4.4) to arbitrary one-forms. Explicitly, we have

$$\langle [\alpha, \beta]_{T^*M}, X \rangle = \langle \alpha, [\pi, \langle \beta, X \rangle]_S \rangle - \langle \beta, [\pi, \langle \alpha, X \rangle]_S \rangle - [\pi, X]_S(\alpha, \beta), \quad (4.7)$$

where $\alpha, \beta \in \Omega^1(M)$, X is a smooth vector field and $[-, -]_S$ on the right-hand side is the Schouten-Nijenhuis bracket of multivector fields on M .

When quantizing the dual of a Lie algebra, we used the convolution algebra on the integrating Lie group. It is thus natural to expect that we can define a quantization of (the dual of) a Lie algebroid by using the convolution algebra on the integrating Lie groupoid. The fact that not all Lie algebroids can be integrated merely corresponds to the fact that not all Poisson manifolds can be quantized with this method. The idea of using Lie groupoids in quantization goes back to work by Karasëv, Weinstein and Zakrzewski, see [6] and references therein. Bonechi *et al.* have recently addressed the problem of quantizing $\mathbb{C}P^1$ to the Podleś sphere with this method [33].

A subtle issue in quantization using groupoids is the introduction of a polarization. Here we will follow the approach proposed by Hawkins [6], which we now briefly describe.

We start from a Poisson manifold (M, π) , which gives rise to the corresponding Lie algebroid T^*M as described above. Instead of integrating T^*M as a Lie algebroid, we will find an *integrating symplectic Lie groupoid* for M , i.e., we construct a Lie groupoid Σ with base M , where Σ is a symplectic manifold with symplectic structure ω . We demand that π is a Poisson map² and that ω is multiplicative. It can be shown [34] that the existence of an integrating symplectic Lie groupoid for M is equivalent to the integrability of the Lie algebroid T^*M . Let us briefly spell out the definition of multiplicative forms on Σ : Consider the set of composable arrows $\Sigma_{[2]} \subset \Sigma \times \Sigma$ in Σ . We then have projection maps $\text{pr}_1, \text{pr}_2 : \Sigma_{[2]} \rightarrow \Sigma$ onto the first and second arrow in the composable pair:

$$\text{pr}_1(g, h) = g \quad \text{and} \quad \text{pr}_2(g, h) = h. \quad (4.8)$$

A form α on Σ is called *multiplicative* if

$$\text{pr}_1^* \alpha + \text{pr}_2^* \alpha = m^* \alpha. \quad (4.9)$$

Having found an integrating symplectic groupoid for the Poisson manifold we wish to quantize, we prequantize Σ as a symplectic manifold in the second step, i.e., we construct a prequantum line bundle (L, h, ∇) over Σ such that $F_\nabla = -2\pi i \omega$.

Third, we have to endow Σ with a groupoid polarization, i.e., we have to introduce a smooth integrable lagrangian distribution that reduces the sections of L over Σ . In our examples, a groupoid polarization is either chosen to be given by Kähler polarization or it is derived from the fact that the base M is a lagrangian submanifold in the integrating symplectic Lie groupoid Σ .

In the final step, we construct a polarized convolution algebra of Σ , using e.g. a Haar system of measures, which is twisted by a $U(1)$ -valued 2-cocycle. In the cases where there is a globally defined symplectic potential on Σ , the cocycle twist essentially encodes the failure of the potential to be multiplicative; in general its role is to induce a multiplication on the fibres of $L \rightarrow \Sigma$. This twisted polarized convolution algebra is then identified with the algebra of functions on the quantization of M .

It should be stressed that there are existence and uniqueness issues remaining open in many of the steps of this quantization procedure. One advantage of this approach is that it avoids the explicit construction of the usual Hilbert space, yielding the quantized algebra of functions rather

²In some treatments it is also demanded that π is an anti-Poisson map. We do not impose this restriction here.

directly. This fact might help generally in the quantization of 2-plectic manifolds, where we expect non-associative structures which cannot be modeled on linear spaces.

4.3 Groupoid quantization of \mathbb{R}^2

The simplest example of a groupoid quantization is that of the real vector space $M = \mathbb{R}^2$ with Poisson structure given by a constant bivector field $\pi = \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, $i, j = 1, 2$. As integrating Lie groupoid, we choose the cotangent bundle $\Sigma = T^*M = M \times M^*$, which we coordinatize by (x^i, p_i) . The symplectic structure on Σ is the natural one, $\omega = dx^i \wedge dp_i$. The groupoid maps are given by the diagram

$$x^i + \frac{1}{2} \pi^{ij} p_j \xrightarrow{(x^i, p_i)} x^i - \frac{1}{2} \pi^{ij} p_j, \quad (4.10)$$

so that we have

$$s(x^i, p_i) = (x^i + \frac{1}{2} \pi^{ij} p_j) \quad \text{and} \quad t(x^i, p_i) = (x^i - \frac{1}{2} \pi^{ij} p_j), \quad (4.11)$$

and we recognize the Bopp shifts familiar from canonical quantization of \mathbb{R}^2 . One readily checks that t is indeed a Poisson map, i.e.,

$$\{t^* f, t^* g\}_\omega = t^* \{f, g\}_\pi. \quad (4.12)$$

It remains to verify the multiplicativity of ω . Consider the composition of arrows

$$x^i + \frac{1}{2} \pi^{ij} (p_j + p'_j) \xrightarrow{(x^i + \frac{1}{2} \pi^{ij} p_j, p'_i)} x^i + \frac{1}{2} \pi^{ij} (p_j - p'_j) \xrightarrow{(x^i - \frac{1}{2} \pi^{ij} p'_j, p_i)} x^i - \frac{1}{2} \pi^{ij} (p_j + p'_j). \quad (4.13)$$

Therefore, $\Sigma_{[2]}$ can be identified with $M \times M^* \times M^*$, which we coordinatize by (x^i, p_i, p'_i) and we have the structure maps

$$\begin{aligned} \text{pr}_1(x^i, p_i, p'_i) &= (x^i + \frac{1}{2} \pi^{ij} p_j, p'_i), \quad \text{pr}_2(x^i, p_i, p'_i) = (x^i - \frac{1}{2} \pi^{ij} p'_j, p_i), \\ \text{and } m(x^i, p_i, p'_i) &= (x^i, p_i + p'_i). \end{aligned} \quad (4.14)$$

It is easy to verify that $\text{pr}_1^* \omega + \text{pr}_2^* \omega = m^* \omega$, and the groupoid Σ is thus indeed an integrating symplectic groupoid for M .

The prequantization of (Σ, ω) is now straightforward: Since ω is exact, we take L to be the trivial line bundle over Σ with the obvious hermitian metric h and connection ∇ such that $F_\nabla = -2\pi i \omega$.

A globally defined symplectic potential for ω is given by $\vartheta = -x^i dp_i$, and as polarization we choose the vector fields in $T\Sigma$ that are in the kernel of ϑ . This reduces functions on Σ to functions on M^* . The twist element σ_π encodes the failure of ϑ to be multiplicative, and we have

$$-i \sigma_\pi^{-1} d\sigma_\pi := (\text{pr}_1^* + \text{pr}_2^* - m^*) \vartheta = d(-\frac{1}{2} p_i \pi^{ij} p'_j), \quad (4.15)$$

which yields

$$\sigma_\pi(p, p') = e^{-\frac{i}{2} p_i \pi^{ij} p'_j}. \quad (4.16)$$

Together with the natural translation-invariant measure on M^* , we thus obtain the twisted convolution product on polarized functions on Σ given by

$$(\tilde{f} \otimes_\pi \tilde{g})(p) = \int_{M^*} dp' \sigma_\pi(p', p - p') \tilde{f}(p') \tilde{g}(p - p'), \quad (4.17)$$

which is the usual Moyal product after Fourier transformation to momentum space.

The groupoid quantization of the two-dimensional torus T^2 is completely analogous, except that one now has to invoke a Bohr-Sommerfeld quantization condition on the leaves of the polarization on the cotangent groupoid $\Sigma \cong T^2 \times \mathbb{R}^2$, which reproduces the usual deformation of Fourier series on T^2 [6].

Analogously, one can quantize κ -Minkowski space, where the integrating groupoid is the cotangent bundle $\Sigma = T^*G$, with G the Lie group of the κ -Minkowski Lie algebra; the cocycle twist here is trivial. Berezin-Toeplitz quantization of Kähler manifolds also fits into this framework; the integrating groupoid here is the pair groupoid, and again the cocycle twist is trivial.

5. Quantization of loop spaces

5.1 2-plectic manifolds, gerbes and loop spaces

As discussed above, a symplectic manifold (M, ω) with symplectic form ω that satisfies the quantization condition $[\omega] \in H^2(M, \mathbb{Z})$ comes naturally with a prequantum line bundle, i.e., there is a line bundle $L \rightarrow M$ with connection ∇ such that $F_\nabla = -2\pi i \omega$. Analogously, a 2-plectic manifold (M, ϖ) with 2-plectic form ϖ satisfying $[\varpi] \in H^3(M, \mathbb{Z})$ comes naturally with a prequantum abelian gerbe, i.e., an abelian gerbe with 2-connection whose curvature is $H = -2\pi i \varpi$.

Since a gerbe may be thought of as a categorification of a vector bundle, an obvious approach to the quantization of 2-plectic manifolds is to categorify conventional quantization, see e.g. [35, 36]. We will report on progress in categorifying Hawkins' groupoid approach in future work.

Here we will follow a different approach: We can map the prequantum gerbe on the 2-plectic manifold (M, ϖ) to a prequantum line bundle on the loop space³ of M using a *transgression map*. Consider the double fibration in which the correspondence space $\mathcal{L}M \times S^1$ is simultaneously fibred over a manifold M and its free loop space $\mathcal{L}M := \mathcal{C}^\infty(S^1, M)$:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ ev \swarrow & & \searrow pr \\ M & & \mathcal{L}M \end{array} \quad (5.1)$$

The projection pr is the obvious one, and the map ev is the evaluation $x(\tau)$ of a loop $x \in \mathcal{L}M$ at a given angle $\tau \in S^1$. We can now construct a map $\mathcal{T} : \Omega^{n+1}(M) \rightarrow \Omega^n(\mathcal{L}M)$ by pulling back an $n+1$ -form α on M along ev and projecting it down to $\mathcal{L}M$ by integrating over the circle fibres. The latter map amounts to filling up one slot of the form with the tangent vector to the loop. Explicitly, the transgression of a form $\alpha = \frac{1}{(n+1)!} \alpha_{\mu_1 \dots \mu_{n+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n+1}}$ is given by

$$(\mathcal{T}\alpha)(x) := ((pr)_! \circ ev^*)\alpha(x) = \oint d\tau \frac{1}{n!} \alpha_{\mu_1 \dots \mu_n \mu_{n+1}}(x(\tau)) \dot{x}^{\mu_{n+1}}(\tau) \delta x^{\mu_1}(\tau) \wedge \dots \wedge \delta x^{\mu_n}(\tau), \quad (5.2)$$

where $\dot{x}^\mu(\tau) = \frac{d}{d\tau} x^\mu(\tau)$ denotes the tangent vector to the loop at the angle τ . The transgression map is a chain map: It maps closed forms to closed forms and exact forms to exact forms. There is therefore a well-defined restriction $\mathcal{T} : H^{n+1}(M, \mathbb{Z}) \rightarrow H^n(\mathcal{L}M, \mathbb{Z})$. More generally, for $\alpha \in$

³When referring to loop space, we will usually mean loops modulo reparametrization invariance.

$\Omega^{n+1}(M)$, $n \geq 0$, one has $\delta \mathcal{T} \alpha = \mathcal{T} d\alpha$, where $d = dx^\mu \frac{\partial}{\partial x^\mu}$ and $\delta = \oint d\tau \delta x^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)}$ denote the exterior derivatives on M and $\mathcal{L}M$, respectively; in particular, $\mathcal{T} df = 0$ for all $f \in \mathcal{C}^\infty(M)$. The transgression of a form is also invariant under reparametrizations of the loop. The map \mathcal{T} is not surjective, and there are more line bundles over $\mathcal{L}M$ than gerbes on M . The inverse map, called “regression”, is generally defined only on the image of \mathcal{T} .

This point of view has been successfully used in lifting the ADHMN construction of magnetic monopoles to a construction of self-dual strings in M-theory [37, 38].

Let us consider the explicit example $M = \mathbb{R}^3$ with 2-plectic form $\varpi = \frac{1}{3!} \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$. Transgressing this form yields a symplectic 2-form on the loop space $\mathcal{L}\mathbb{R}^3$ given by

$$\omega = \mathcal{T} \varpi = \oint d\tau \frac{1}{2} \varepsilon_{ijk} \dot{x}^k(\tau) \delta x^i(\tau) \wedge \delta x^j(\tau). \quad (5.3)$$

Note that the kernel of ω is non-trivial, as

$$\iota_{X_\alpha}(\mathcal{T} \varpi) = 0 \quad \text{for } X_\alpha = \oint d\rho \alpha(\rho) \dot{x}^i(\rho) \frac{\delta}{\delta x^i(\rho)}. \quad (5.4)$$

The vector field X_α , however, generates reparametrization transformations and it is not contained in the space of derivations acting on functions of reparametrization invariant loops in $\mathcal{L}\mathbb{R}^3$. Restricting to such loops, we can therefore either invert the components of ω or, equivalently, follow the usual construction via Hamiltonian vector fields to arrive at the Poisson bracket

$$\{f, g\}_\omega := \oint d\tau \oint d\rho \delta(\tau - \rho) \pi^{ijk} \frac{\dot{x}_k(\rho)}{|\dot{x}(\rho)|^2} \left(\frac{\delta}{\delta x^i(\tau)} f \right) \left(\frac{\delta}{\delta x^j(\rho)} g \right) \quad (5.5)$$

on $\mathcal{C}^\infty(\mathcal{L}\mathbb{R}^3)$. This expression for the Poisson bracket is reparametrization invariant.

5.2 Groupoid quantization of $\mathcal{L}\mathbb{R}^3$

We now come to the quantization of the loop space $\mathcal{L}M$ for $M = \mathbb{R}^3$ using the groupoid approach. Up to a few subtleties, we can follow closely the discussion of the case \mathbb{R}^2 . As integrating groupoid, we take $\Sigma = T^* \mathcal{L}M \cong \mathcal{L}T^*M$. There are two points about this identification that we have to stress. First of all, we exclude elements $\alpha = \oint d\tau \alpha_i(\tau) \delta x^i(\tau)$ of $T^* \mathcal{L}M$ that have coefficients $\alpha_i(\tau)$ which are distributional. Second, the identification is only local and does not respect reparametrization invariance. Thus, if we coordinatize elements of $\mathcal{L}T^*M$ by pairs $(x^i(\tau), p_i(\tau))$, then the momenta $p_i(\tau)$ are not invariant under reparametrizations of the loop but transform like the velocity vectors $\dot{x}^i(\tau)$. This will be visible in all formulas below.

Comparison with the case \mathbb{R}^2 suggests introducing the source and target maps

$$s(x^i(\tau), p_i(\tau)) = x^i(\tau) + \frac{1}{2} \pi^{ijk} p_j(\tau) \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}, \quad t(x^i(\tau), p_i(\tau)) = x^i(\tau) - \frac{1}{2} \pi^{ijk} p_j(\tau) \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}, \quad (5.6)$$

which are local in the loop parameter and reparametrization invariant. The set of composable arrows is $\Sigma_{[2]} = \mathcal{L}(M \times M^* \times M^*)$, and the projections and multiplication maps are

$$\begin{aligned} \text{pr}_1(x^i(\tau), p_i(\tau), p'_i(\tau)) &:= (x^i(\tau) + \tfrac{1}{2} \pi^{ijk} p_j(\tau) \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}, p'_i(\tau)) , \\ \text{pr}_2(x^i(\tau), p_i(\tau), p'_i(\tau)) &:= (x^i(\tau) - \tfrac{1}{2} \pi^{ijk} p'_j(\tau) \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}, p_i(\tau)) , \\ \text{m}(x^i(\tau), p_i(\tau), p'_i(\tau)) &:= (x^i(\tau), p_i(\tau) + p'_i(\tau)) . \end{aligned} \quad (5.7)$$

The natural symplectic structure on Σ reads as

$$\omega = \oint d\tau \oint d\rho \, \delta(\tau - \rho) \delta x^i(\tau) \wedge \delta p_i(\rho) , \quad (5.8)$$

and ω can be derived from the symplectic potential

$$\vartheta = \oint d\rho \, x^i(\rho) \delta p_i(\rho) . \quad (5.9)$$

Note that both loop space forms ω and ϑ are reparametrization invariant.

It turns out that the target map t is a Poisson map only to lowest order in π , i.e.,

$$\begin{aligned} \{\text{t}^* f, \text{t}^* g\}_\omega &:= \oint d\tau \left(\left(\frac{\delta}{\delta x^i(\tau)} \text{t}^* f \right) \left(\frac{\delta}{\delta p_i(\tau)} \text{t}^* g \right) - \left(\frac{\delta}{\delta x^i(\tau)} \text{t}^* g \right) \left(\frac{\delta}{\delta p_i(\tau)} \text{t}^* f \right) \right) \\ &= \text{t}^* \{f, g\}_\pi + \mathcal{O}(\pi^2) \quad \text{for } f, g \in \mathcal{C}^\infty(\mathcal{L}\mathbb{R}^3) . \end{aligned} \quad (5.10)$$

Similarly, the symplectic form ω is multiplicative only to lowest order in π : $(\text{pr}_1^* + \text{pr}_2^* - \text{m}^*)\omega = \mathcal{O}(\pi)$. Higher corrections to the groupoid structure can be computed order by order in π . Here we will just develop the quantization to lowest order.

We now follow the usual remaining steps of the groupoid approach to quantization. To pre-quantize (Σ, ω) , we introduce a trivial line bundle L over Σ together with a connection ∇ such that $F_\nabla = -2\pi i \omega = -2\pi i \mathcal{T} \vartheta$. The polarization is again induced by the symplectic potential, i.e., we consider a distribution in the kernel of ϑ given by (5.9). This polarization renders functions independent of $x(\tau)$, but note that it preserves the dependence on $\dot{x}(\tau)$. The latter point is related to the problem of introducing polarizations on 2-plectic manifolds.

Following the algorithm, we now introduce a twist element from the failure of ϑ to be a multiplicative one-form,

$$(\text{pr}_1^* + \text{pr}_2^* - \text{m}^*)\vartheta = -i \sigma_\pi^{-1} \delta \sigma_\pi , \quad (5.11)$$

and obtain

$$\sigma_\pi(p_i(\rho), p'_j(\tau)) = \exp \left(-\tfrac{i}{2} p_i(\rho) \pi^{ijk} \dot{x}_k(\tau) p'_j(\tau) \delta(\rho - \tau) \right) . \quad (5.12)$$

It is difficult to introduce the notion of a twisted polarized convolution algebra on the groupoid Σ , as it is not clear how to define a reparametrization invariant system of Haar measures on Σ . However, by analogy with the case \mathbb{R}^2 , we conclude that the quantization of Σ yields a coordinate algebra on loop space that is deformed as

$$[\hat{x}^i(\tau), \hat{x}^j(\rho)] = \tfrac{i}{2} \pi^{ijk} \widehat{\frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}} \delta(\tau - \rho) . \quad (5.13)$$

The expression (5.13) agrees via transgression with a quantization of one-forms as suggested by the Poisson-like brackets developed in [20]. In general, with $\omega = \mathcal{T}\varpi$, the transgression of (2.8) yields

$$\mathcal{T}\{\alpha, \beta\}_{\varpi}^{(1)} = \{\mathcal{T}\alpha, \mathcal{T}\beta\}_{\omega}, \quad (5.14)$$

and so maps the problem of quantizing one-forms on M to quantizing functions on loop space. Since $\mathcal{T}d\iota_{x_\alpha}\beta = 0$, the Jacobi identity is recovered. Moreover, it agrees with results of M-theory computations presented in [18, 19] using Dirac quantization of open membrane dynamics on an M5-brane. In [19, 39], corrections to (5.13) in higher orders of π have been computed. It might be interesting to compare these corrections to the corrections which arise from adjusting the groupoid structure to higher orders in π .

Again, an analogous groupoid quantization can be applied to the loop space of the three-dimensional torus T^3 , with interesting topological features similar to those arising in closed string sigma-models with T^3 target space. In particular, the algebra (5.13) agrees with Lüster's algebra (2.9) via the identification $p_i \sim \dot{x}_i$. Moreover, the algebra (5.13) agrees essentially with (2.12) after integrating over τ and ρ .

6. Conclusions and outlook

In this article we reviewed Hawkins' groupoid approach to quantization. This approach is particularly suited for higher quantization, as it circumvents the introduction of the usual Hilbert space. We demonstrated how the groupoid approach can be applied to the quantization of 2-plectic manifolds via the detour through loop spaces: The prequantum gerbe on a 2-plectic manifold yields a prequantum line bundle on the corresponding loop space, which can be subjected to ordinary quantization. In making the transition to loop space, we traded the difficulties in dealing with gerbes for the problems of working with infinite-dimensional manifolds. Moreover, the expected nonassociativity in the gerbe picture has been resolved through associative structures acting on non-separable Hilbert spaces. The deformed coordinate algebra on loop space that we found via this method agrees with earlier M-theory results and anticipated closed string nonassociativity.

A number of interesting open questions, however, still remain: First of all, one should examine if it is possible to follow the other, more natural approach to the quantization of 2-plectic manifolds by categorifying the groupoid approach. Second, it would be useful to study the ensuing higher bracket structures and compare them to those that have already been studied in the literature, see e.g. [40] and references therein. Finally, once a satisfying quantization of S^3 is obtained, it would be interesting to use the resulting quantum algebra in the Basu-Harvey equation (2.4) and ultimately in the various M2-brane models that have been studied over the last four years. We expect that this would contribute to the solution of some of the as yet open problems of these models.

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References

- [1] N. M. J. Woodhouse, *Geometric quantization*, Oxford mathematical monographs, Clarendon, New York, 1992.
- [2] Y. Nambu, *Generalized Hamiltonian dynamics*, *Phys. Rev.* **D7** (1973) 2405–2414.
- [3] L. Takhtajan, *On foundation of the generalized Nambu mechanics*, *Commun. Math. Phys.* **160** (1994) 295–316, [hep-th/9301111].
- [4] P. Gautheron, *Some remarks concerning Nambu mechanics*, *Lett. Math. Phys.* **37** (1996) 103–116.
- [5] J. DeBellis, C. Saemann, and R. J. Szabo, *Quantized Nambu-Poisson manifolds and n -Lie algebras*, *J. Math. Phys.* **51** (2010) 122303, [arXiv:1001.3275].
- [6] E. Hawkins, *A groupoid approach to quantization*, *J. Symplectic Geom.* **6** (2008) 61–125, [math.SG/0612363].
- [7] J. DeBellis, C. Saemann, and R. J. Szabo, *Quantized Nambu-Poisson manifolds in a 3-Lie algebra reduced model*, *JHEP* **04** (2011) 075, [arXiv:1012.2236].
- [8] C. Saemann and R. J. Szabo, *Branes, quantization and fuzzy spheres*, *PoS CNCFG* **2010** (2010) 005, [arXiv:1101.5987].
- [9] C. Saemann and R. J. Szabo, *Quantization of 2-plectic manifolds*, arXiv:1106.1890. Proc. of the 4-th Annual Meeting of the European Non-Commutative Geometry Network, Bucharest, 25.4.–30.4. 2011.
- [10] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B503** (1997) 220–238, [hep-th/9608163].
- [11] D. Tsimpis, *Nahm equations and boundary conditions*, *Phys. Lett.* **B433** (1998) 287–290, [hep-th/9804081].
- [12] R. C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022, [hep-th/9910053].
- [13] P. S. Howe, N. D. Lambert, and P. C. West, *The self-dual string soliton*, *Nucl. Phys.* **B515** (1998) 203–216, [hep-th/9709014].
- [14] A. Basu and J. A. Harvey, *The M2-M5 brane system and a generalized Nahm’s equation*, *Nucl. Phys.* **B713** (2005) 136–150, [hep-th/0412310].
- [15] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030, [hep-th/9903205].
- [16] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032, [hep-th/9908142].
- [17] C.-S. Chu and D. J. Smith, *Towards the quantum geometry of the M5-brane in a constant C-field from multiple membranes*, *JHEP* **04** (2009) 097, [arXiv:0901.1847].
- [18] E. Bergshoeff, D. S. Berman, J. P. van der Schaar, and P. Sundell, *A noncommutative M-theory five-brane*, *Nucl. Phys.* **B590** (2000) 173–197, [hep-th/0005026].
- [19] S. Kawamoto and N. Sasakura, *Open membranes in a constant C-field background and noncommutative boundary strings*, *JHEP* **07** (2000) 014, [hep-th/0005123].
- [20] J. C. Baez, A. E. Hoffnung, and C. L. Rogers, *Categorified symplectic geometry and the classical string*, *Commun. Math. Phys.* **293** (2010) 701–725, [arXiv:0808.0246].

- [21] D. Lüst, *T-duality and closed string non-commutative (doubled) geometry*, *JHEP* **12** (2010) 084, [arXiv:1010.1361].
- [22] R. Blumenhagen and E. Plauschinn, *Nonassociative gravity in string theory?*, *J. Phys.* **A44** (2011) 015401, [arXiv:1010.1263].
- [23] R. Blumenhagen, A. Deser, D. Lust, E. Plauschinn, and F. Rennecke, *Non-geometric fluxes, asymmetric strings and nonassociative geometry*, *J. Phys.* **A44** (2011) 385401, [arXiv:1106.0316].
- [24] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, *Commun. Math. Phys.* **225** (2002) 33–66, [hep-th/0101219].
- [25] P. Bouwknegt, K. Hannabuss, and V. Mathai, *Nonassociative tori and applications to T-duality*, *Commun. Math. Phys.* **264** (2006) 41–69, [hep-th/0412092].
- [26] I. Ellwood and A. Hashimoto, *Effective descriptions of branes on non-geometric tori*, *JHEP* **12** (2006) 025, [hep-th/0607135].
- [27] P. Grange and S. Schafer-Nameki, *T-duality with H-flux: Noncommutativity, T-folds and $G \times G$ structure*, *Nucl. Phys.* **B770** (2007) 123–144, [hep-th/0609084].
- [28] P. Bouwknegt, K. Hannabuss, and V. Mathai, *C^* -algebras in tensor categories*, *Clay Math. Proc.* **12** (2011) 127–165, [arXiv:math.QA/0].
- [29] C. Iuliu-Lazaroiu, D. McNamee, and C. Saemann, *Generalized Berezin quantization, Bergman metrics and fuzzy Laplacians*, *JHEP* **09** (2008) 059, [arXiv:0804.4555].
- [30] M. A. Rieffel, *Lie group convolution algebras as deformation quantizations of linear Poisson structures*, *Am. J. Math.* **112** (1990) 657–685.
- [31] V. Kathotia, *Kontsevich’s universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, *Int. J. Math.* **11** (2000) 523–551, [math.QA/9811174].
- [32] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, Cambridge University Press (2005).
- [33] F. Bonechi, N. Ciccoli, N. Staffolani, and M. Tarlini, *The quantization of the symplectic groupoid of the standard Podleś sphere*, arXiv:1004.3163.
- [34] M. Crainic and R. L. Fernandes, *Integrability of Poisson brackets*, *J. Diff. Geom.* **66** (2004) 71, [math.DG/0210152].
- [35] J. C. Baez and C. L. Rogers, *Categorified symplectic geometry and the string Lie 2-algebra*, *Homology Homotopy Appl.* **12** (2010) 221–236, [arXiv:0901.4721].
- [36] C. L. Rogers, *2-plectic geometry, Courant algebroids, and categorified prequantization*, arXiv:1009.2975.
- [37] C. Saemann, *Constructing self-dual strings*, *Commun. Math. Phys.* **305** (2011) 513–532, [arXiv:1007.3301].
- [38] S. Palmer and C. Saemann, *Constructing generalized self-dual strings*, *JHEP* **10** (2011) 008, [arXiv:1105.3904].
- [39] D. S. Berman and B. Pioline, *Open membranes, ribbons and deformed Schild strings*, *Phys. Rev.* **D70** (2004) 045007, [hep-th/0404049].
- [40] J. A. de Azcarraga and J. M. Izquierdo, *n -ary algebras: A review with applications*, *J. Phys.* **A43** (2010) 293001, [arXiv:1005.1028].